

Polynomial Optimization (401-3908-21G)

Adam Kurpisz

ETH Zürich

ONLINE LECTURES - SET UP

- ▶ Lectures via Zoom, link on Moodle (problems—restart).
- ▶ Lectures are recorded, recordings on Moodle.
- ▶ You are muted, unmuting \Rightarrow agreement to be on recording.
- ▶ Chat function, set to private mode, asking question anonymous.

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DATES AND TIMES

Volume: 3h/week—type G

Wednesday, 16-18 — lectures

Friday, 13-14 — exercise sessions, Federica Cecchetto

Office hours, 16-18 — office hours, please register one day before

- ▶ 10 minutes break, if you do not agree please complain now.
- ▶ Type G— single group in exercise sessions.
- ▶ Any deviation will be communicated: Friday 26.02 — lecture

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PERFORMANCE ASSESSMENT

Type: end-of-semester examination

- ▶ Exam will take place between Monday, May 24 (calendar week 21) up until and including Friday, June 18 (calendar week 24)

Repetition of the exam:

- ▶ Possible to repeat without having to re-enroll in a course.
- ▶ A repetition date, in first two weeks of the Fall semester.
- ▶ The repetition may not be used as alternative for the first attempt.
- ▶ Students must register for such a repetition date using myStudies.
- ▶ Possible once the result of their first try has been officially announced.

Legal basis: <https://ethz.ch/content/dam/ethz/common/docs/weisungssammlung/files-en/end-of-semester-examinations.pdf>

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Exam:

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- ▶ Similar to problem sets/exercises in the script.
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CONTINUOUS PERFORMANCE ASSESSMENT

Type: c. Learning tasks

- ▶ Not mandatory, possible grade upgrade.

Exercise session:

- ▶ Volunteers can present solutions in the exercise sessions.
 - ▶ Can improve the total course unit grade by up to 0.25 grade points.
 - ▶ Students can still achieve the maximum grade of 6 with the exam only.
 - ▶ 2 smaller or 1 big exercise is enough.
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- ▶ Solutions to exercises can be uploaded and corrected, but no grade bonus is offered.

Legal basis: [https:](https://www.ethz.ch/content/dam/ethz/common/docs/weisungssammlung/files-en/directive-continuous-performance-assessment.pdf)

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MATERIALS

Script:

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Notes:

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Formally no prerequisites.

Desirable: background in

- ▶ linear programming
- ▶ integer programming

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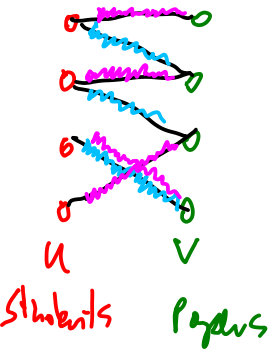
- ▶ linear programming
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LP

LP is a very a powerful technique.

Input: $G(U \cup V, E)$

Output: max cardinality matching



WHAT?

Question: *Is there life beyond LP methods?*

WHAT?

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Goal: Broad exposition to mathematical optimization methods.

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LP

Mathematical Optimization

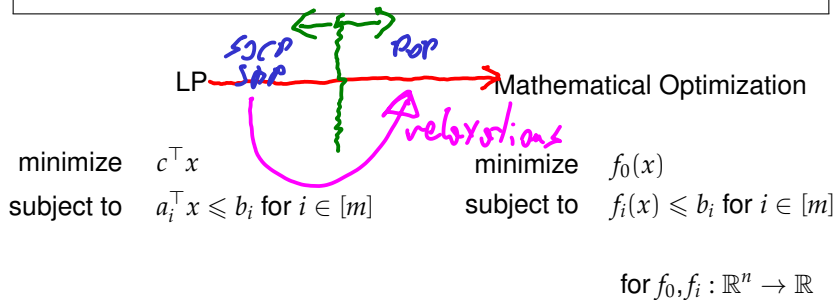
minimize $c^\top x$

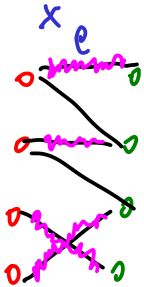
subject to $a_i^\top x \leq b_i$ for $i \in [m]$

WHAT?

Question: *Is there life beyond LP methods?*

Goal: Broad exposition to mathematical optimization methods.





$$Ax \leq b$$

↓
TV

Challenges for LP

- ▶ LP relaxation might not be tight
- ▶ Problems might just have nonlinear nature

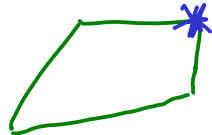
WHY?

$$\max \sum_{e \in E} x_e$$

$$\sum_{e \in \delta(v)} x_e \leq 1 \quad \forall v \in V$$

$$x_e \in [0, 1] \quad \forall e \in E$$

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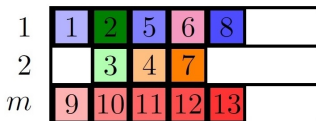
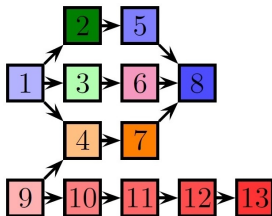
WHY?

Challenges for LP

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WHY?

Scheduling with precedence constraints.

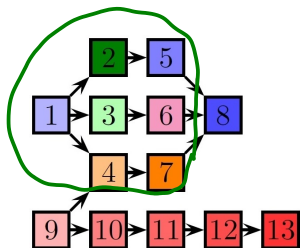


► Garey, Johnson **OPEN8**

► Greedy

WHY?

Scheduling with precedence constraints.



1	1	2	5	6	8	
2		3	4	7		
m	9	10	11	12	13	

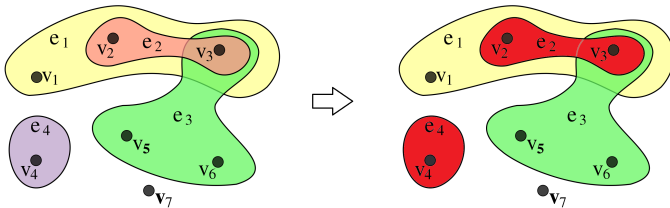
► Garey, Johnson **OPEN8**

① ② ③ ④ ... ⑬

► Greedy

WHY?

Hypergraph matching

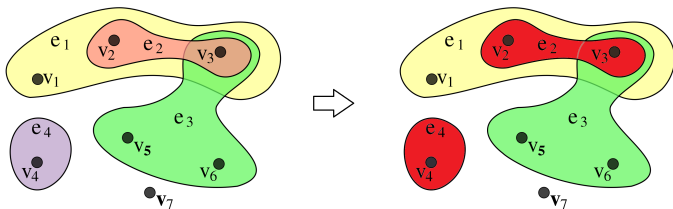


▶ $\frac{k+1+\epsilon}{3}$ -approximation

▶ Local Search

WHY?

Hypergraph matching



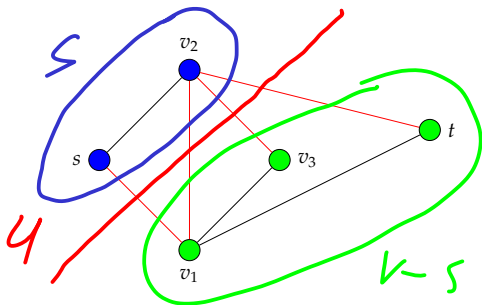
▶ $\frac{k+1+\epsilon}{3}$ -approximation

▶ Local Search

$$\frac{k+1}{2}$$

WHY?

Max-Cut problem



Input: $G(V, E)$

Output:

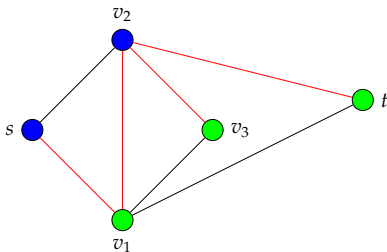
$S \subseteq V:$

$\text{cut}(S, V-S)$
is max.

- ▶ ≈ 0.878 -approximation
- ▶ Mathematical Programming

WHY?

Max-Cut problem



-
- ▶ ≈ 0.878 -approximation

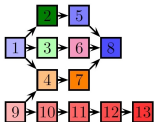
SDP

- ▶ Mathematical Programming

WHY?

Do we really need LP and mathematical optimization methods?

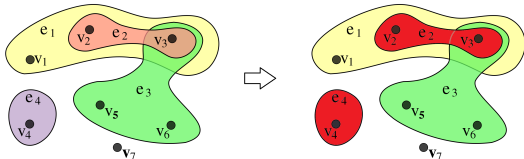
OPTIMIZATION PROBLEMS



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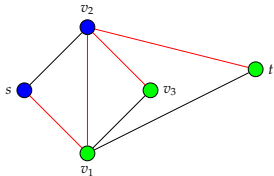
► Garey, Johnson **OPEN8**

► Greedy



► $\frac{k+1+\epsilon}{3}$ -approximation

► Local Search



► ≈ 0.878 -approximation

► Math. Prog.

OPTIMIZATION PROBLEMS



Garey, Johnson **OPEN8**

Greedy



$\frac{k+1+\epsilon}{3}$ approximation

Local Search



≈ 0.878 -approximation

Math. Prog.

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Do we really need LP and mathematical optimization methods?

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They are much easier to use than tailor made algorithms.

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They are much easier to use than tailor made algorithms.

	Single User	Floating	Server Single Socket	Server Dual Socket	Server Quad Socket
<u>AMPL</u>	\$4000	\$6000	\$8000	\$14000	\$24000
<i>Linear-quadratic solvers:</i>					
<u>CPLEX</u>	\$9500	\$14500	<i>Contact us for details</i>		
<u>Gurobi</u> *	\$10000	\$20000	<i>Contact us for details</i>		
<u>Xpress</u>	\$8000	\$12000	\$16000	\$32000	\$64000

Intro 1

LP-1
CP-1

WHAT?

17:18

LP/CP



SOCP



SDP



POP ↔ Nonnegativity

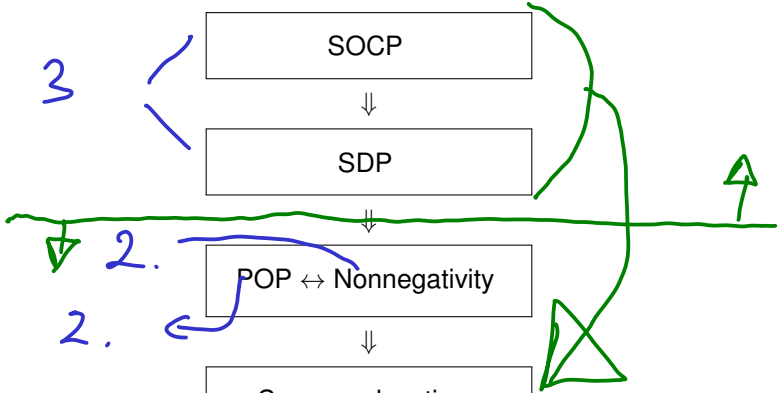


Convex relaxations
(SOCP, SDP)

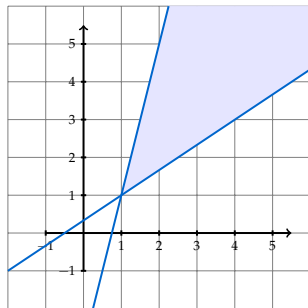
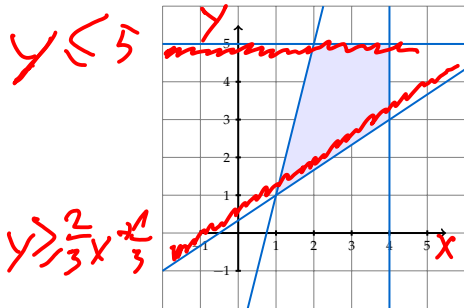
3

2.

2.



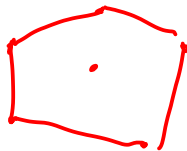
INEQUALITY VS VERTEX REPRESENTATION



Inequality representation

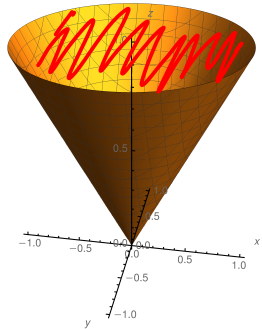
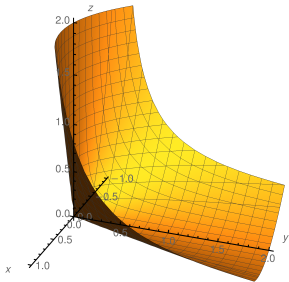
Vertex representation

minimize $c^T x$
 subject to $a_i^T x \leq b_i$ for $i \in [m]$



SoCP

Lorentz cone.



APPLICATION

Markowitz portfolio optimization

Input: n assets, budget B , predicted change in price $p \in \mathbb{R}^n$. $p_i = 2$

Output: Portfolio of assets maximizing the profit.

Input: n assets, budget B , predicted mean change in price $\bar{p} \in \mathbb{R}^n$, covariance matrix Σ .

Output: Portfolio of assets minimizing the risk (variance), for expected profit at least r .

$$\begin{aligned}
 & \max x^T p \\
 & \mathbb{1}^T x = B \quad \sum_{i \in \{n\}} x_i = B \\
 & x_i \geq 0 \quad \forall i \in \{n\} \\
 & x_i \leq \frac{B}{10}
 \end{aligned}$$

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$$\min x^T \Sigma x$$

$$\bar{p}^T x \geq r$$

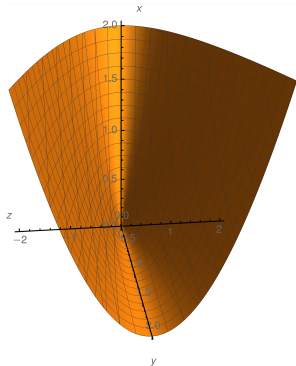
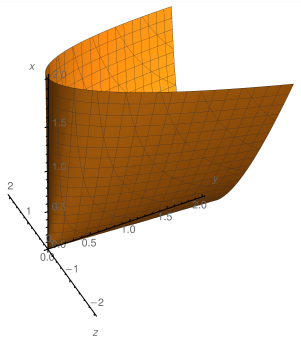
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$$x_i \geq 0 \quad \forall i \in \{n\}$$

QP
SOCP

SDP

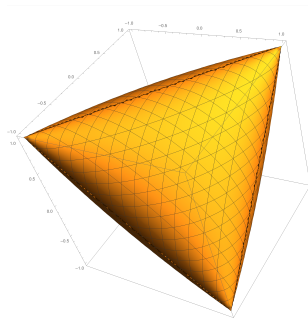
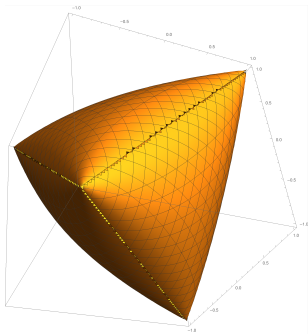
Cone of PSD matrices.



$$\{(x, y, z) \in \mathbb{R}^3 : \begin{bmatrix} x & z \\ z & y \end{bmatrix} \succeq 0\}$$

SDP

Eliptop



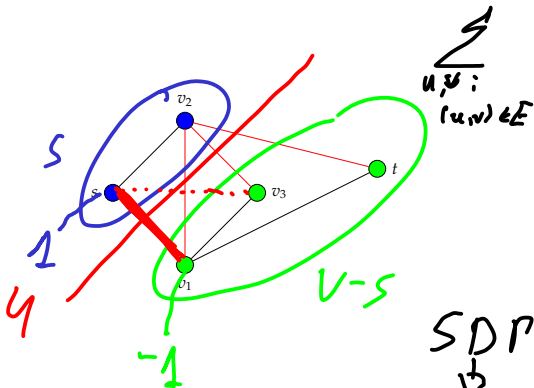
$$\left\{ X \in \mathbb{R}^6 : \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12} & X_{22} & X_{23} \\ X_{13} & X_{23} & X_{33} \end{bmatrix} \succeq 0, X_{11} = X_{22} = X_{33} = 1 \right\}$$

APPLICATION

Max-Cut problem

Input: Undirected graph $G(V, E)$.

Output : Cut $S \in V$ that maximizes size of the cut $S, V \setminus S$.



$$\sum_{u, v: (u, v) \in E}$$

$$\frac{1}{2} (1 - x_u x_v)$$

1 if $(u, v) \in (S, V \setminus S)$
0 o.w.

$$x_v \in \{-1, 1\} \forall v \in V$$

SDP
↓
0.878

LP
↓
0.5



POP

Polynomial Optimization

minimize $f_0(x)$
subject to $f_i(x) \leq b_i$ for $i \in [m]$

for $f_0, f_i \in \mathbb{R}[x_1, \dots, x_n]$

Captures:

- ▶ All the combinatorial problems listed above.
- ▶ SOCP
- ▶ SDP
- ▶ Many others

POP

Polynomial Optimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i \text{ for } i \in [m] \end{array}$$

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Captures:

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$$x_i \in \{0, 1\}$$

$$\left. \begin{array}{l} x_i^2 - x_i \geq 0 \\ x_i^2 - x_i \leq 0 \end{array} \right\} \Rightarrow$$

$$x_i^2 - x_i = 0$$

$$x_i = 1 \quad \text{or} \quad x_i = 0$$

POP

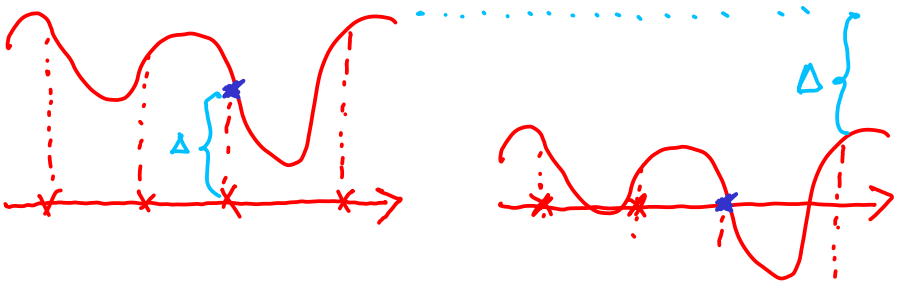
Where is a catch?

It is not convex.

POP

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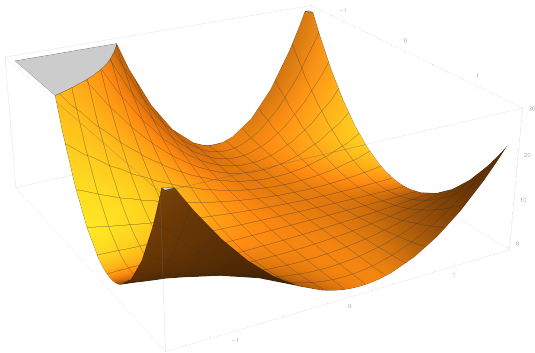


CONNECTION TO NONNEGATIVITY

Is $9x^2y^2 - 6x^2y + x^2 - 6xy^2 + 8xy - 2x + y^2 - 2y + 1$ nonnegative?

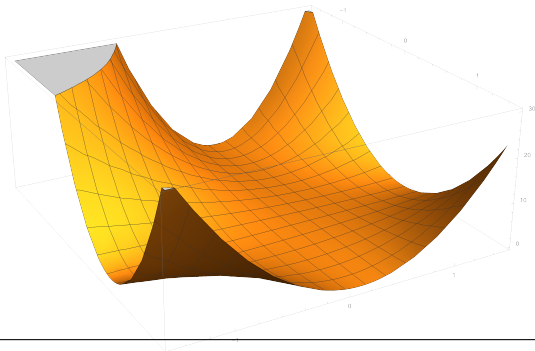
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$$(-3xy + x + y - 1)^2 \geq 0 \quad \forall x, y \in \mathbb{R}$$

HILBERT'S 17TH QUESTION

Is every nonnegative polynomial an SoS polynomial?

$$\text{Motzkin: } M(x, y) := x^4y^2 + x^2y^4 - 3x^2y^2 + 1$$

Is every nonnegative polynomial an SoS of rational functions?

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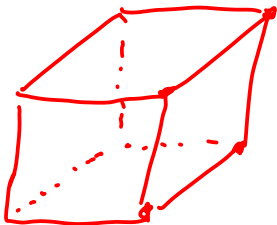
Is every nonnegative polynomial an SoS of rational functions?

Artin 27

YES

POSITIVESTELLENSATZ

Good news: Every polynomial nonnegative on a “nice” subset of \mathbb{R}^n has SoS certificate of nonnegativity.



$$x_i \in [0, 1]$$

Moreover: It can be found efficiently using Semidefinite programming.

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Is SoS the only option?

We will look for other families of nonnegative polynomials such that:

- ▶ Efficient optimization over their cone is possible
- ▶ Provide “tight” relaxations for Polynomial Optimization Problems.

Polya Positivstellensatz: Optimization free technique.

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Combinatorial optimization

- ▶ Min $s - t$ Cut
- ▶ Max Cut
- ▶ scheduling with precedence constraints

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- ▶ Stability criteria

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- ▶ Variants of Markowitz portfolio optimization

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- ▶ Have script during the lectures.
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- ▶ 0.25 grade points for active participation.
- ▶ Attend, working systematically is the key to success.
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